Stochastic Views on Diamond Search Model: Asymmetrical Cycles and Fluctuations

Masanao Aoki
Department of Economics
University of California, Los Angeles
Fax Number 310-825-9528, e-mail aoki@econ.ucla.edu
and
Yoshimasa Shirai
Department of Economics
Keio University
Mita, Tokyo
e-mail yshirai@econ.keio.ac.jp

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Abstract

This paper abandons one of the simplifying assumptions in the standard formulation of search models in the economic literature by assuming that the number of economic agents in the model is large but finite. By explicitly treating the fraction of the number of agents of one type (employed, say) as a random variable rather than a deterministic number, the paper simplifies the analysis associated with the analysis of cycles, and calculates variances of the fraction as well.

We derive the probability distribution of the fraction by solving a (backward) Chapman-Kolmogorov equation, i.e., the master equation and solved it approximately in terms of power series expansion in terms of $N^{-1/2}$ where N is the number of agents in the model.

We show that in models with several locally stable equilibria asymmetrical cycles exist, and derives variances of the fraction about locally stable equilibria in stationary states. What is important to notice is the ease with which cycles can be established and fluctuations characterized when the number of participants are kept finite at the beginning and later let it approach infinity if desired, rather than assuming it to be infinite from the beginning.

Key words: Search model, Master equations, Fokker-Planck equations, Fluctuations, Asymmetrical cycles

1 Introduction

The search model of Diamond (1982), and its elaboration in Diamond and Fudenberg (1989) have been influential, as evidenced by frequent citations in the search literature. By casting their model in a setting with an infinitely many agents, however, their dynamic analysis is necessarily deterministic, and totally abstracted from fluctuations of the fraction, which could be substantial and important in real life.

We re-examine their model in a framework of a large but a finite number of agents.[1] We have two objectives in recasting the original model this way: One to obtain information on fluctuations about the equilibria, and the other to provide a simpler explanation than Diamond and Fudenberg for cyclical behavior. Dynamic behavior of the model is now described by the backward Chapman-Kolmogorov equation, or what is called the master equation in the physics and ecology literature.[2] It describes how the probabiity for the fraction evolves with time. This master equation is then approximately solved to yield two equations: one is an ordinary differential equation for the average or expected value of the fraction of the employed. The other is a partial differential equation, known as the Fokker-Planck equation, for random deviations of the fraction about the mean. When we let the number of agents go to infinity the equation for the mean reproduces the equation for the fraction derived by Diamond. The Fokker-Planck equation is new. The critical points of the ordinary differential equation, and the endogenously determined reservation cost expression jointly yield information on the equilibria, and asymmetrical cyclical behavior.

Our approach allows us to draw more natural conditions than in Diamond and Fudenberg under which the model exhibits asymmetric cyclical behavior similar to business cycles. Fluctuations about aggregate dynamics occur in our analysis because micro-shocks intrinsic in our models do not vanish when the number of agents in the model is finite. In our setting arrivals of production and trading opportunities are stochastic. With positive probabilities net effects do not vanish but accumulate to change the fraction of employed from one basin of attraction to the other.[3]

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[1] This example was suggested by J.M. Orszak as one of his comments on Chapter 5 in Aoki (1996).

[2] See Aoki (1996, Sec. 5.1) or van Kampen (1992, p.97) for the source of this name.

[3] The idea of micro-shock creating aggregate risk is pointed out by Jovanovic (1987). His main point is that in the nonlinear systems micro-shocks intrinsic in the model do not vanish. Kirman (1993) discusses a mechanism of stochastic cycles. The focus of his model is on herding effects. See Aoki (1998, p.436) for comparison of our method and that of Kirman. Furthermore, our model in this paper has optimizing agents. In such a model fluctuations among two basins of attractions are still possible.

1.1 Model

There is a large but a finite number, N, of agents,[4] who are in one of two possible states, employed and unemployed. Of the N agents in the model, n of them are employed, and $N - n$ are unemployed. The state of the collection of the N agents is n, or equivalently the fraction $e = n/N$. Each of the $N - n$ unemployed persons independently encounters a production opportunity which appears at the rate of $a\Delta t$ in a small time interval Δt. If the opportunity is accepted, it yields the unit output and at the cost c, where c is a nonnegative random number with a known distribution function G. There is a reservation or threshold cost $c^*(n)$, to be determined endogenously below, above which the opportunity is rejected as being too costly. When the opportunity is accepted, the person's status changes from being unemployed to employed. Each of n employed persons independently encounters a trading opportunity at the rate $b(n/N)$ per unit time. When an employed person encounters a trading opportunity he forms a pair with another randomly selected employed person, and the pair trade and each of the pair consumes the output of the partner to receive instantaneous utility v and their status changes to being unemployed from employed. See Diamond (1982) for some explanations for these assumptions.

Let $W_e(n, t)$ be the present discounted value of lifetime utility of an employed person, and let $W_u(n, t)$ be that of an unemployed when the state is n. Because n is a random variable in this paper, we take the expectation of these random value functions later after we derive the stationary distribution of n. We drop t from the argument of the value functions because dynamic programing involves infinite horizon and the problem is time-homogeneous. The value functions are evaluated in Sec. 5 below after we discuss the dynamics for the mean of the fraction and a Fokker-Planck equation for the fluctuations about the mean in Section 3 and 4. In the next Section 2 we discuss the transition rates of the underlying stochastic process. Section 7 discusses first passage times between two locally stable equilibria when the model dynamics have multiple locally stable equilibria. The paper concludes with Section 8.

2 Transition Rates

We model the problem as a jump Markov process. Thus, the model is completely specified by the transition rates whch describe movements of agents over a small interval of time.

To an unemployed agent production opportunities arrrive at the rate a as a Poisson process. Each production opportunity if undertaken yields a unit of output with cost c. Only production with cost c^* or less will be undertaken.

[4]We take N to be a large fixed number. It is straightforward to let N be random. See Kelly (1979) for example.

The transition rate from n to $n+1$ is given by $(N-n)aG(c^*)$, where c^* is the "reservation" cost in the sense that only the production with cost $c \leq c^*$ is undertaken. Since this reservation cost is a choice variable and depends on n/N we write it as $c^*(n/N)$ or as $c^*(n)$ for short in the following.

For an employed agent trading opportunities arrive as a Poisson process at the rate $\beta(n/N)$. His probability for being one of the random pair is $1 - C_{n-1,2}/C_{n,2} = 2/n$. We define the arrival rate of trading opportunity for an agent to be $b(n/N) := (2/n)\beta(n/N)$. While an employed agent waits for a trading partner, the probability is $[C_{n-1,2}/C_{n,2}]\beta = [(n-2)/2]\beta$ that a pair involving other employed agents trade, thus decreasing n to $n-2$. In aggregate, then, the transition rate from state n to $n-2$ is given by $(n/2)b(n/N)$.

3 Aggregate Dynamics: Dynamics for the Mean of the Fraction

The master equation (see Aoki (1996, Sec. 5.1)) is

$$dp_n(t)/dt = r_{n-1}p_{n-1}(t) + l_{n+2}p_{n+2}(t) - (r_n + l_n)p_n(t),$$

with obvious boundary conditions imposed at n at 0 and N, and near these values as shown in Section 5.

From our prvious discussion

$$r_n = (N-n)aG(c^*(n/N)) = N(1-e)aG(c^*(n/N)),$$

and

$$l_n = \frac{n}{2}b(\frac{n}{2}).$$

Since this equation cannot be solved exactly we proceed as in Aoki (1996, p. 123)to derive an approximate solution. Change variables as

$$\frac{n}{N} = \phi + \frac{\xi}{\sqrt{N}}.$$

The variable ϕ is the expected fraction of employed and ξ represents random fluctuations about the mean. This scaling implies that fluctuations are expected to be of the order of$\sqrt{N}$.[5] In this change of variables, note that $(n+1)/N = \phi + N^{-1/2}(\xi + 1/\sqrt{N})$, and so on. For example , ξ changes by $2/\sqrt{N}$ in l_{n+2}. Let $\Pi(\xi, t) := p_n(t)$. The master equation is now rewritten in terms of Π by noting that

$$dp_n(t)/dt = \partial\Pi/\partial t + (\partial\Pi/\partial\xi)(d\xi/dt),$$

[5] That this is the correct order is indicated by the fact that the coefficients of the Fokker-Planck equation for ξ, to be derived below, are independent of N.

where

$$d\xi/dt = -\sqrt{N} d\phi/dt.$$

In the Taylor series expansion, after subsituting the change of variables, we match the left-hand side of the order $\sqrt{N}$ with the terms of the same order on the right-hand side. We derive the aggregate dynamic equation for ϕ as

$$\frac{d\phi}{dt} = \Phi(\phi) := (1-\phi)aG(c^*) - \phi b(\phi). \tag{1}$$

This is in agreement with the dynamic equation for e in Diamond, his (1). Here we define $\Phi(\phi)$ as above as short-hand because this grouping of terms arise several times below.

4 Dynamics for the Fluctuations

The rest of terms are for determining the distribution of ξ. By collecting terms of order $O(N^0)$ in the Taylor series expansion this equation is seen to be given by

$$\partial\Pi/\partial t = A\Pi + A\xi\partial\Pi/\partial\xi + C\partial^2\Pi/\partial\xi^2 + O(N^{-1/2}), \tag{2}$$

with

$$A = -\Phi'(\phi) = aG(c^*) + b(\phi) + \phi b'(\phi) - a(1-\phi)G'[c^*(\phi)](c^*)'(\phi),$$

and

$$C = \frac{1}{2}(1-\phi)aG(c^*) + \phi b(\phi).$$

This is a type of Fokker-Planck equation which can be solved as discussed in Aoki (1996, Sec. 5.13), for example. As we discuss shortly, the local equilibria of the dynamics are the zeros of the function Φ. Its derivative Φ' is negative at those local equilibria which are locally asymptotically stable, i.e., at those locally stable equilibria A is positive. Note that the coefficient $C = 2\phi b(\phi)$ is at the critical points. Eq.(2) can be solved by the method of separation of variables. Let $\Pi(\xi, t) = T(t)X(\xi)$. Then we obtain

$$T'(t)/T(t) = A + A\xi X'(\xi)/X(\xi) + CX''(\xi)/X(\xi) = -\theta,$$

where θ is some constant.

To obtain a stationary solution, set θ to zero. Rewriting the equation for X as

$$(C/A)X'' + (\xi X)' = 0.$$

In the case where $X'(0) = 0$, we can solve it as

$$X(\xi) = X(0)\exp(-\frac{A}{C}\frac{\xi^2}{2}).$$

We have thus shown that this statioanry distribution for ξ is normally distributed with mean zero and variance C/A. Its variance is given by

$$var(\xi) = \frac{C}{A}.$$

With two or more locally stable equilibria, the probability mass around each of the critical points may overlap and assign positive probability to the neighboring critical points. This is one sufficient condition for fluctuations to spill over to the neighboring basins of attractions. Even if this does not happen, we show later that expected first passage times from one basin to the neighboring ones are finite, i.e., cycles are possible.

5 Value Functions

Denote the discount rate by r. Value functions depend on the fraction n/N rather than on n directly. For shorter notation, however, we denote them by $W_e(n)$ and $W_u(n)$ for the employed and unemployed when the number of the employed is n. For an employed agent, we obtain the relation for the value functions as

$$\begin{aligned} rW_e(n) &= b(n/N)[v + W_u(n-2) - W_e(n)] + (N-n)aG(c^*(n))[W_e(n+1) - W_e(n)] \\ &\quad + \frac{n-2}{2} b(n/N)[W_e(n-2) - W_e(n)] \end{aligned}$$

for n between 3 and $N-1$, and for an umemployed agent[6]

$$\begin{aligned} rW_u(n) &= a\int_0^{c^*(n)} [W_e(n+1) - W_u(n) - z]dG(z) + (N-n-1)aG(c^*(n))[W_u(n+1) \\ &\quad - W_u(n))] + \frac{n}{2} b(n/N)[W_u(n-2) - W_u(n)], \end{aligned}$$

for $n = 2, 3, \ldots, N-1$. There are boundary relations which we do not use, but mention here for completeness:

$$rW_e(N) = b(1)[v + W_u(N-2) - W_e(N)] + \frac{N-2}{2} b(1)[W_e(N-2) - W_e(N)],$$

and

$$rW_e(n) = (N-n)aG^*(n)[W_e(n+1) - W_e(n)],$$

for $n = 1, 2$, where $G^*(n) := G(c^*(n))$. Finally

$$rW_u(n) = aG^* + (N-n-1)aG^*(n)[W_u(n+1) - W_u(n)],$$

[6]The probability intensity for the transition from state n to state $n+1$ is $(N-n)aG^*$, where G^* is short-hand for $G(c^*)$, of which $(N-n-1)aG^*$ is the intensity for other unemployed agents to become employed, while he remains unemployed. The intensity for him to become employed is aG^*.

for $n = 0, 1$.

We next take the expected values of these value functions with respect to the stationary distributions of n. By changing variables as done in Sec. 3 we show below that the stationary distribution for ξ is normally distributed with mean zero and variance which is a function of ϕ but is independent of N. This is a posteriori justification for the change of variables we have performed.

Rather than obtaining optimal sequences of the reservation costs from the set of equations displayed above, we first derive the expressions for the expected values of the value functions and then derive the expression for the reservation costs as functions of ϕ up to terms of order $O(1/N)$.

5.1 Expected Value Functions

In this section we take the expected values of the value functions earlier derived.

Change variables as indicated earlier, and define

$$V_u(\phi + \xi/\sqrt{N}) := W_u(n),$$

and

$$V_e(\phi + \xi/\sqrt{N}) := W_e(n).$$

The expressions in the square brackets become

$$W_u(n+1) - W_u(n) = V_u(\phi+\xi/\sqrt{N}+1/N) - V_u(\phi+\xi/\sqrt{N}) = \frac{1}{N}V_u'(\phi) + o(1/N),$$

and

$$W_u(n-2) - W_u(n) = -\frac{2}{N}V_u'(\phi) + o(1/N),$$

respectively.

Noting that $E\xi = 0$, and $E\xi^2 = \sigma^2$, the expected value function becomes, after dropping terms of the order $1/N$ or less we arrive at

$$rV_u(\phi) = aG^*[V_e(\phi) - V_u(\phi)] - a\hat{c} + \Phi(\phi)V_u'(\phi). \tag{3}$$

Proceeding analogously, and dropping terms of the order $O(1/N)$ or smaller

$$rV_e(\phi) = b(\phi)[v + V_u(\phi) - V_e(\phi)] + \Phi(\phi)V_e'(\phi). \tag{4}$$

Details of algebra is in Appendix.

These two equations correspond with (4a) and (4b) in Diamond and Fudenberg.

Making use of the fact that $\Phi(\phi_e) = 0$, where ϕ_e denote locally stable equilibrium points, (3) and (4) yield equilibrium value functions

$$V_e(\phi_e) = \frac{(r + aG^*)b(\phi_e)v - ab(\phi_c)\hat{c}}{r[r + b(\phi_e) + aG^*]},$$

and

$$V_u(\phi_e) = \frac{ab(\phi_e)G^*v - a(r + b(\phi_e))\hat{c}}{r[r + b(\phi_e + aG^*]},$$

where G^* and $\hat{c}$ are evaluated at ϕ_e.

As they point out, by subtracting (5) from (4), and setting $c^*(\phi) = V_e(\phi) - V_u(\phi)$, the reservation cost is given implicitly by

$$rc^* = b(v - c^*) - a\int^{c^*}(c^* - c)dG(c) + \Phi(V_e' - V_u'),$$

where the last term is recognized as $dc^*/dt) = (dc^*/dt)(d\phi/dt)$.

Thus, the analogy with the case of infinite number of agents holds.

We can actually see that this choice of c^* is optimal by differentiating the expected value functions with respect to c^*, noting that $b(\phi)$ is exogenously specified and its derivative with respect to c^* is zero. Solving for the derivatives of the expected value functions with respect to c^* we see that they are both zero. This is the first order condition for optimality. The second order condition may be shown to hold by taking derivatives once more.

6 Multiple Equilibria and Cycles: An Example

To give the basic idea behind construction of models with several equilibria here is an example with two locally stable equilibria.

We take $b(\phi) = a\phi$ to simplify algebra. We also let r/a be denoted by r, that is we normalize both b and r by a. Suppose that there are two possible costs: $0 = c_1 \leq c_2$, that is the distribution function $G(c)$ is a step function; $G(c_1) = p > 0; G(c_2) = 1$. The right-hand side of the dynamics for the aggregate equation (1) is either $\Phi_1(\phi) = a[(1-\phi)p - \phi^2]$, or $\Phi_2(\phi) = a[1 - \phi - \phi^2]$ depending on the range of the argument ϕ. We show below that Φ_1 applies when ϕ is not greater than ψ, and above it Φ_2 prevails.

There are thus two critical points. They are the roots of $\Phi_i(\phi) = 0$, $i = 1, 2$, and are given by

$$\phi_1 = [\sqrt{p^2 + p} - p]/2,$$

and

$$\phi_2 = [\sqrt{5} - 1]/2 := \kappa.$$

We see that $\Phi_i'(\phi_i)$ are negative for $i = 1, 2$, that is, the critical points are locally stable.

From the optimality condition, $c_1^* = c^*(\phi_1)$ is determined by

$$rc_1^* = \phi_1(v - c_1^*) - pc_1^*,$$

or

$${c_1}^* = \frac{\phi_1 v}{r + \phi_1 + p},$$

if $0 < c_1^* < c_2$.

The second value $c_2^* = c^*(\phi_2)$ is determined by

$$rc_2^* = \phi_2(v - c_2^*) - c_2^* + c_2(1-p),$$

or

$$c_2^* = \frac{\phi_1 v + c_2(1-p)}{r + \phi_2 + 1},$$

if $c_2 < c_2^*$.

The two basins of attractions are separated at

$$\psi = \frac{c_2(r+p)}{(v - c_2)},$$

where we assume that $v > c_2$, that is the value of Φ undergoes a discontinous change at this value:

$$\Phi_1(\psi-) < 0,$$

and

$$\Phi_2(\psi+) > 0.$$

See Fig. 1 Therefore, if there is a large positive disturbance near ϕ_1 which makes the variable ϕ to cross the boundary at ψ, then the derivative is positive and the disturbance is amplified and ϕ is attracted to ϕ_2. Conversely a large negative disturbance near ϕ_2 will cause the state variable to be attracted to ϕ_1.

The conditions to ensure $0 \leq \phi_1 < \psi < \phi_2 \leq 1$ are

$$c_2 < \frac{v\kappa}{r + p + \kappa},$$

and

$$\sqrt{p^2 + p}/2 < p/2 + c_2(r+p)/(v_2 - c_2).$$

Thus a small p and not too large c_2 will suffice to satisfy these conditions.

The same construction works with three critical points although conditions on the parameters are more complicated to state. The critical points are determined as functions of $p_1 = G(0)$ and $p_1 + p_2 = G(c_2)$, where $0 = c_1 < c_2$. At $c_3 > c_2$ $G(c_3) = 1$. We ensure that $c_i < c^*(\phi_i) < c_{i+1}$ holds, $i = 1, 2$, and $c_3 < c_3^*$.

We now have possibilities of the total of three cycles; between ϕ_1 and ϕ_2,ϕ_2 and ϕ_3, but also between ϕ_1 and ϕ_3.

7 Concluding Discussion

We have re-examined the Diamond search model for the case of a finite number of agents, that is, by not assuming an infinite number of agents from the beginning. With a finit number of agents, the fraction of employed agents is a random variable which fluctuates about its mean value. We have derived that the fluctuations are of the form $\xi/\sqrt{N}$ where N is the total number of agents, and the distribution function for ξ is Gaussian with mean and finite variance which is a function of the expected fraction of the employed agents. We have shown via a simple example that the model can have several locally stable equilibria and that the fraction of the employed agents may fluctuate between the pair of equilibria. We have shown that this leads to a simpler explanations of asymmetrical cycles, among others.

8 Reference

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Appendix

The Master equation is given by,

$$\begin{aligned}\frac{dP_n(t)}{dt} &= (N-n+1)aG(c^{n-1})P_{n-1}(t) + \frac{n+2}{2}b\left(\frac{n+2}{N}\right)P_{n+2}(t)\\ &\quad -\left[(N-n)aG(c^n) + \frac{n}{2}b\left(\frac{n}{N}\right)\right]P_n(t)\end{aligned}$$

c^n can be considered as a function for n/N. Hence we can write $c^n = c(n/N)$. Moreover, we have $n/N = \phi + \xi/\sqrt{N}$. Also, setting $P_n(t) = \Pi(\xi, t)$ and through Taylor expansions of functions $b(\cdot)$, $c(\cdot)$, and $\Pi(\xi, t)$ we can rewrite the master equation as below,

$$\frac{d\Pi(\xi,t)}{dt}\left(= \frac{\partial \Pi}{\partial t} - \frac{\partial \Pi}{\partial \xi}\frac{d\phi}{dt}\sqrt{N}\right)$$

$$
\begin{aligned}
= \; & aN\left(1-\phi-\frac{\xi}{\sqrt{N}}+\frac{1}{N}\right)\left[G(c)+G'(c)c'\left(\frac{\xi}{\sqrt{N}}-\frac{1}{N}\right)\right]\left[\Pi-\frac{\partial\Pi}{\partial\xi}\frac{1}{\sqrt{N}}+\frac{1}{2}\frac{\partial^2\Pi}{\partial\xi^2}\frac{1}{N}\right] \\
& +\frac{N}{2}\left(\phi+\frac{\xi}{\sqrt{N}}+\frac{2}{N}\right)\left[b(\phi)+b'(\phi)\left(\frac{\xi}{\sqrt{N}}+\frac{2}{N}\right)\right]\left[\Pi+\frac{\partial\Pi}{\partial\xi}\frac{2}{\sqrt{N}}+\frac{\partial^2\Pi}{\partial\xi^2}\frac{2}{N}\right] \\
& -aN\left(1-\phi-\frac{\xi}{\sqrt{N}}\right)\left(G(c)+G'(c)c'\frac{\xi}{\sqrt{N}}\right)\Pi \\
& -\frac{N}{2}\left(\phi+\frac{\xi}{\sqrt{N}}\right)\left(b(\phi)+b'(\phi)\frac{\xi}{\sqrt{N}}\right)\Pi+O(N^{-1}) \\
= \; & aG(c)\Pi+NaG'c'\left(1-\phi-\frac{\xi}{\sqrt{N}}+\frac{1}{N}\right)\left(\frac{\xi}{\sqrt{N}}-\frac{1}{N}\right)\Pi \\
& -aNG'c'\left(1-\phi-\frac{\xi}{\sqrt{N}}\right)\frac{\xi}{\sqrt{N}}\Pi \\
& -aNG\left(1-\phi-\frac{\xi}{\sqrt{N}}+\frac{1}{N}\right)\left(\frac{\partial\Pi}{\partial\xi}\frac{1}{\sqrt{N}}-\frac{1}{2}\frac{\partial^2\Pi}{\partial\xi^2}\frac{1}{N}\right) \\
& -aNG'c'\left(1-\phi-\frac{\xi}{\sqrt{N}}+\frac{1}{N}\right)\left(\frac{\xi}{\sqrt{N}}-\frac{1}{N}\right)\left(\frac{\partial\Pi}{\partial\xi}\frac{1}{\sqrt{N}}-\frac{1}{2}\frac{\partial^2\Pi}{\partial\xi^2}\frac{1}{N}\right) \\
& +b(\phi)\Pi+\frac{Nb'}{2}\left(\phi+\frac{\xi}{\sqrt{N}}+\frac{2}{N}\right)\left(\frac{\xi}{\sqrt{N}}+\frac{2}{N}\right)\Pi-\frac{Nb'}{2}\left(\phi+\frac{\xi}{\sqrt{N}}\right)\frac{\xi}{\sqrt{N}}\Pi \\
& +\frac{Nb}{2}\left(\phi+\frac{\xi}{\sqrt{N}}+\frac{2}{N}\right)\left(\frac{\partial\Pi}{\partial\xi}\frac{2}{\sqrt{N}}+\frac{\partial^2\Pi}{\partial\xi^2}\frac{2}{N}\right) \\
& +\frac{Nb'}{2}\left(\phi+\frac{\xi}{\sqrt{N}}+\frac{2}{N}\right)\left(\frac{\xi}{\sqrt{N}}+\frac{2}{N}\right)\left(\frac{\partial\Pi}{\partial\xi}\frac{2}{\sqrt{N}}+\frac{\partial^2\Pi}{\partial\xi^2}\frac{2}{N}\right)+O(N^{-1}) \\
= \; & -\{(1-\phi)aG-\phi b\}\frac{\partial\Pi}{\partial\xi}\sqrt{N}+\{[aG(c)+b+\phi b'-(1-\phi)aG'c']\,\Pi \\
& +[aG+b+\phi b'-(1-\phi)aG'c']\,\xi\frac{\partial\Pi}{\partial\xi}+\left(\frac{1-\phi}{2}aG+\phi b\right)\frac{\partial^2\Pi}{\partial\xi^2}\}N^0+O(N^{-1/2})
\end{aligned}
$$

The comparison of ther term with order $\sqrt{N}$ gives us the aggregate law of motion for ϕ.

$$\frac{d\phi}{dt}=(1-\phi)aG(c(\phi))-\phi b(\phi).$$

The Fokker-Planck equation is the coefficient of the term with order N^0 and is given by,

$$A\Pi+A\frac{\partial\Pi}{\partial\xi}+C\frac{\partial^2\Pi}{\partial\xi^2}=\theta.$$

where $A=aG(c)+b+\phi b'-(1-\phi)aG'c'$ and $C=(1-\phi)aG/2+\phi b$.

Next, rewrite the system of equations for W_e, W_u and c in terms of average of n/N, ϕ. With new notations for value functions introduced in section 4, the equation W_e can be expanded as follows;

$$
\begin{aligned}
& r\cdot\left(V_e(\phi)+V_e'(\phi)\frac{\xi}{\sqrt{N}}\right)\\
&= \left(b+b'\frac{\xi}{\sqrt{N}}\right)\left[b+V_u+V_u'\cdot\left(\frac{\xi}{\sqrt{N}}-\frac{2}{N}\right)-V_e-V_e'\frac{\xi}{\sqrt{N}}\right]\\
&\quad +aN\left(1-\phi-\frac{\xi}{\sqrt{N}}\right)\left(G(c^*)+G'(c^*)c^{*\prime}\frac{\xi}{\sqrt{N}}\right)\left[V_e+V_e'\cdot\left(\frac{\xi}{\sqrt{N}}+\frac{1}{N}\right)-V_e-V_e'\frac{\xi}{\sqrt{N}}\right]\\
&\quad +\frac{N}{2}\left(\phi+\frac{\xi}{\sqrt{N}}-\frac{2}{N}\right)\left(b+b'\frac{\xi}{\sqrt{N}}\right)\left[V_e+V_e'\cdot\left(\frac{\xi}{\sqrt{N}}-\frac{2}{N}\right)-V_e-V_e'\frac{\xi}{\sqrt{N}}\right]
\end{aligned}
$$

where $c^*=c^*(\phi)$ and $c^{*\prime}=c^{*\prime}(\phi)$. Taking expectations of above over ξ, we get the following equation (note that $\xi=0$ and ξ^2 are zero and σ^2 respectively);

$$
\begin{aligned}
rV_e &= b(v+V_u-V_e)+((1-\phi)aG-\phi b)V_e'\\
&\quad +[b'V_u'-(aG'c^{*\prime}+2b')V_e']\frac{\sigma^2}{N}+\frac{2b\cdot(V_e'-V_u')}{N}.
\end{aligned}
$$

Similarly, for W_u, we get

$$
\begin{aligned}
& r\cdot\left(V_u(\phi)+V_u'(\phi)\frac{\xi}{\sqrt{N}}\right)\\
&= a\left[V_e+V_e'\left(\frac{\xi}{\sqrt{N}}+\frac{1}{N}\right)-V_u-V_u'\frac{\xi}{\sqrt{N}}\right]\left(G+G'c^{*\prime}\frac{\xi}{\sqrt{N}}\right)-a\int_{\underline{c}}^{c^*+c^{*\prime}\xi/\sqrt{N}} zdG(z)\\
&\quad +\left(1-\phi-\frac{\xi}{\sqrt{N}}-\frac{1}{N}\right)a\left(G+G'c^{*\prime}\frac{\xi}{\sqrt{N}}\right)V_u'-\left(\phi+\frac{\xi}{\sqrt{N}}\right)\left(b+b'\frac{\xi}{\sqrt{N}}\right)V_u'
\end{aligned}
$$

Hence,

$$
\begin{aligned}
rV_u &= aG(V_e-V_u)-a\hat{c}+[(1-\phi)aG-\phi b]V_u'\\
&\quad +[aG'c^{*\prime}V_e'-(2aG'c^{*\prime}+b')V_u']\frac{\sigma^2}{N}+\frac{aG\cdot(V_e'-V_u')}{N},
\end{aligned}
$$

where $\hat{c}=\int_{\underline{c}}^{c^*(\phi)} zdG(z)$. For c, we have,

$$
c^*+c^{*\prime}\frac{\xi}{\sqrt{N}}=V_e-V_u+V_e'\cdot\left(\frac{\xi}{\sqrt{N}}+\frac{1}{N}\right)-V_u'\frac{\xi}{\sqrt{N}}.
$$

Again, by taking expectation about ξ we get,

$$
c^*(\phi)=V_e(\phi)-V_u(\phi)+O(N^{-1}).
$$

Now, take the difference between rV_e and rV_u, we get

$$
\begin{aligned}
& r(V_e-V_u)\\
&= b[v-(V_e-V_u)]-aG(V_e-V_u)+a\hat{c}+[(1-\phi)aG-\phi b](V_e'-V_u')\\
&\quad -2(aG'c^{*\prime}+b')(V_e'-V_u')\frac{\sigma^2}{N}+\frac{2b-aG}{N}(V_e'-V_u').
\end{aligned}
$$

Substituting the relationships $c^* = V_e - V_u + O(N^{-1})$ and $c^{*\prime} = V_e' - V_u' + O(N^{-1})$ into above, we get

$$rc^* = b(v - c^*) - a(Gc^* - \hat{c}) + [(1 - \phi)aG(c^*) - \phi b(\phi)]c^{*\prime},$$

where the terms of order less than N^{-1} are omitted here.

The pair $(c^*(\phi), \phi)$ at the critical point is determined by the equations,

$$\begin{aligned} rc^* &= b(\phi)(v - c^*) - a(G(c^*)c^* - \hat{c}) + [(1 - \phi)aG(c^*) - \phi b(\phi)]c^{*\prime} \qquad \text{and} \\ 0 &= (1 - \phi)aG(c^*(\phi)) - \phi b(\phi). \end{aligned}$$

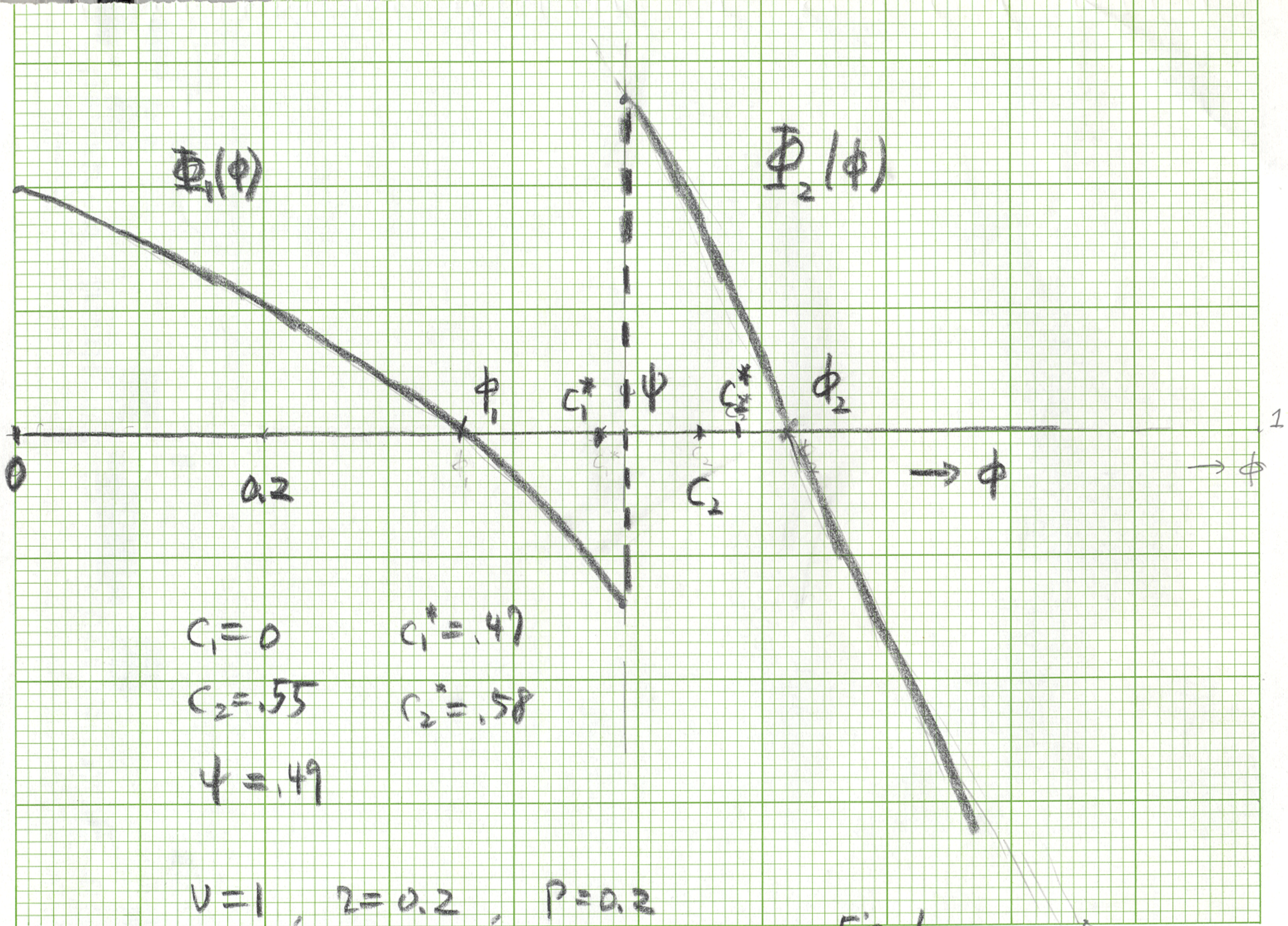
$\Phi_1(\phi)$
$\Phi_2(\phi)$
0
0.2
1
ϕ_1
c_1^*
ψ
c_2^*
ϕ_2
c_2
$\rightarrow \phi$
$c_1 = 0$
$c_1^* = .47$
$c_2 = .55$
$c_2^* = .58$
$\psi = .49$
$v = 1$, $z = 0.2$, $p = 0.2$

Fig. 1